

hep-th/9504004

EPHOU-95-001

April 1995

Multi-Instanton Effect in Two Dimensional QCD

Tetsuyuki Ochiai

ochiai@particle.phys.hokudai.ac.jp

*Department of Physics
Hokkaido University
Sapporo 060 , Japan*

Abstract

We analyze multi-instanton sector in two dimensional $U(N)$ Yang-Mills theory on a sphere. We obtain a contour integral representation of the multi-instanton amplitude and find “neutral” configurations of the even number instantons are dominant in the large N limit. Using this representation, we calculate the 1,2,3,4 bodies interactions and the free energies for $N = 3, 4, 5$ numerically and find that in fact the multi-instanton interaction effect essentially contribute to the large N phase transition discovered by Douglas and Kazakov.

1 Introduction

In the last two years, there is remarkable progress toward understanding relation between large N QCD and string theory. Important contribution to this subject was made by Gross and Taylor[1]. They showed that “two dimensional QCD is a string theory”. Strictly speaking, the partition function of $U(N)$ Yang-Mills theory on Riemann surface Σ_G is equivalent to that of no fold string theory with target space Σ_G , in other words, sum over branched covering maps of Riemann surface Σ_G . But Douglas and Kazakov discovered that on the topology of sphere the system has a 3rd order phase transition at $(\lambda A)_c = \pi^2$ in the large N limit and showed the equivalence is restricted within the strong coupling region $\lambda A > \pi^2$ [2].

Originally the phase transition was discovered from the analysis of so called Cauchy problem in this system. It is well known that when we calculate matrix model partition function in the large N limit, we encounter the Cauchy problem in eigenvalue distribution function. In our case the partition function has the similar structure as the one of Gaussian matrix model and we encounter a two-cut Cauchy problem in $\lambda A > \pi^2$. By exploring the free energy, there is 3rd order gap between the weak and strong coupling region.

One physical explanation of the phase transition is the following. From the string point of view (it is view from one side i.e. strong coupling region), covering map from sphere to sphere (which corresponds to $1/N$ leading term of Yang-Mills free energy on sphere) allows arbitrary but even number of branch points $(2n-2)$ according to arbitrary number of winding (n) . Each branch point has competing entropy (target space area A) and energy. In addition there are many ways to construct world sheet from n target spaces. If the entropy beats the energy, the string expansion becomes infinite and the phase transition occurs [8]. This scenario succeeded to explain the large N phase transition qualitatively and semi-quantitatively (it gives critical coupling $(\lambda A)_c \simeq 11.9$). But this point of view seems to be insufficient because it lacks global perspective and gauge field point of view.

Recently, Gross and Matytsin showed that instanton induces the phase transition [5]. They obtained the following results.

1. Yang-Mills partition function on sphere can be expressed by sum over instantons.
2. The 0 instanton sector gives the weak coupling result.
3. In the weak coupling region the 1 instanton amplitude of charge ± 1 has exponential damping factor and thus is negligible.

In addition, the 1 instanton amplitude of charge ± 1 becomes oscillating in the strong coupling region. But its contribution to the free energy is order $N^{-1/2}$ and thus negligible in the large N limit. Hence it is important to investigate the effect of the multi-instanton.

In this paper we explore the effect of multi-instanton and find out that there is some sort of neutrality in the large N limit. That is, “neutral” configurations of the multi-instanton are dominated in this limit. We also find that the interaction rather than the entropy essentially cause the phase transition. We conjecture 2 bodies force, 4 bodies force, 6 bodies force, \dots equally contribute to the large N phase transition.

The content of this paper is following. In section 2 the multi-instanton amplitude and the partition function are rewritten to multiple contour integrals and the neutrality is found. In section 3 nature of one instanton and two instantons systems are analyzed in detail. In section 4 numerical calculation of the multi-instanton amplitude and the free energy are performed. In appendix using the above formulation, the multi-instanton contribution to Wilson loop is rewritten to multiple contour integrals.

2 Contour integral representation and neutrality

We begin with Migdal-Rusakov heat kernel representation for the Yang-Mills partition function on a sphere[9],

$$\begin{aligned} Z_{QCD} &= \int \mathcal{D}A_\mu \exp\left(-\frac{N}{4\lambda} \int d^2x \sqrt{g} \text{tr} F_{\mu\nu} F^{\mu\nu}\right) \\ &= \sum_R (\dim R)^2 \exp\left(-\frac{\lambda A}{2N} C_2(R)\right), \end{aligned} \quad (1)$$

where A is area of the sphere, R is irreducible representation of the gauge group $U(N)$ and $C_2(R)$ is the value of the second Casimir operator of rep R . Because R is characterized by Young tableau, sum over row lengths in Young tableau replaces sum over representations. We remark that the condition that Young tableau must be stairway-like is irrelevant because $\dim R$ has form of the Van der Monde determinant and thus has permutation symmetry. Then the sum becomes free sum over \mathbf{Z}^N and we can rewrite it by the Poisson resummation formula. We get the following expression¹² [3, 4, 5]:

$$Z_{QCD} = e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} \sum_{\{m_i\} \in \mathbf{Z}^N} e^{-\frac{2\pi^2 N}{A} \sum_{i=1}^N m_i^2} w(\{m\}), \quad (2)$$

$$\begin{aligned} w(\{m\}) &\equiv \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \prod_{i < j}^N (y_{ij}^2 - 4\pi^2 m_{ij}^2) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \\ &= \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta(y_i + 2\pi m_i) \Delta(y_i - 2\pi m_i) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \end{aligned} \quad (3)$$

¹Hereafter we absorb λ into A

²A similar expression was obtained by M. Caselle et al [6]. They showed that the phase transition is due to the winding modes of “fermion on circle”.

where $y_{ij} = y_i - y_j$ and $m_{ij} = m_i - m_j$. In [3, 4, 5] it was showed that $\{m\}$ corresponde to all Euclidean classical solutions (which we call instanton) up to gauge transformation and has nonperturbative effect with respect to both $1/N$ and λ . Hereafter we call m_i instanton charge and call number of nonzero m_i 's instanton number. The $w(\{m\})$ looks like Gaussian matrix model but there is a deformation in the Van der Monde determinant Δ . In this section we rewrite $w(\{m\})$ using the method of ortho-polynomial (in this case Hermite polynomial) and obtain a new multiple contour integrals representation. The new representation makes clear the role of the multi-instanton which was not known.

2.1 0 and 1 instanton sector

In the case of 0 instanton (i.e all $m_i = 0$), $w(\{m\})$ is itself partition function of Gaussian matrix model and has the following form:

$$w(0) = \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta^2(y) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \equiv Z_G\left(\frac{N}{2A}\right) = c_N \left(\frac{N}{2A}\right)^{-\frac{N^2}{2}}. \quad (4)$$

In the case of 1 instanton (i.e. $m_i = m\delta_{ik}$), Using the property of the Van der Monde determinant and the Taylor series expansion of the Hermite polynomial, we obtain,

$$\begin{aligned} w(m) &= \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta(y_i + 2\pi m\delta_{ik}) \Delta(y_i - 2\pi m\delta_{ik}) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \\ &= (N-1)! \sum_{n=0}^{N-1} h_0 \cdots \check{h}_n \cdots h_{N-1} \int_{-\infty}^{\infty} dy_k P_n(y_k + 2\pi m) P_n(y_k - 2\pi m) e^{-\frac{N}{2A} y_k^2} \\ &= \frac{1}{N} Z_G\left(\frac{N}{2A}\right) \sum_{n=0}^{N-1} \sum_{l=0}^n ({}_nC_l)^2 (-4\pi^2 m^2)^{n-l} \frac{h_l}{h_n} \\ &= \frac{1}{N} Z_G\left(\frac{N}{2A}\right) \oint \frac{dt}{2\pi i} e^{-\frac{4\pi^2 m^2 N t}{A}} \left(1 + \frac{1}{t}\right)^N, \end{aligned} \quad (5)$$

where

$$P_n(x) = \frac{1}{2^n \left(\frac{N}{2A}\right)^{\frac{n}{2}}} H_n\left(\sqrt{\frac{N}{2A}} x\right) \quad (6)$$

and

$$\int_{-\infty}^{\infty} dy P_n(y) P_m(y) e^{-\frac{N}{2A} y^2} = h_n \delta_{nm} = \sqrt{2\pi} \left(\frac{A}{N}\right)^{n+\frac{1}{2}} n! \delta_{nm}. \quad (7)$$

This contour integral representation for the 1 instanton amplitude was obtained in [5] from another aspect. They calculated it using large N saddle point method and found it to have behavior change from exponential damping to oscilating at $A = m^2 \pi^2$. The detail is given in section 3.

2.2 multi-instanton sector

In the same way, we can rewrite multi-instanton amplitudes to multiple contour integrals:

$$\begin{aligned}
& w(\{m\}) \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta(y_i + 2\pi m_i) \Delta(y_i - 2\pi m_i) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2} \\
&= \sum_{\mu \in S_N} \text{sgn} \mu \sum_{\sigma \in S_N} \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \prod_{i=1}^N P_{\sigma(i)}(y_i + 2\pi m_i) P_{\mu \circ \sigma(i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2}, \quad (8)
\end{aligned}$$

where S_N is the permutation group on N object.

According to number of nonzero m 's, i.e. number of instantons, the following analysis is different. Assume number of nonzero m 's is k , using permutation symmetry, nonzero m 's are driven to m_i 's from $i=1$ to $i=k$. Then we have,

$$\begin{aligned}
w(\{m\}) &= \sum_{\mu \in S_N} \text{sgn} \mu \sum_{\sigma \in S_N} \prod_{i=k+1}^N h_{\sigma(i)} \delta_{\sigma(i), \mu \circ \sigma(i)} \\
&\quad \times \prod_{i=1}^k \int_{-\infty}^{\infty} dy_i P_{\sigma(i)}(y_i + 2\pi m_i) P_{\mu \circ \sigma(i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2} \\
&= (N-k)! \sum_{\mu \in S_k} \text{sgn} \mu \sum_{a_1 \neq \dots \neq a_k} h_0 \dots h_{a_1} \dots h_{a_k} \dots h_{N-1} \\
&\quad \times \prod_{i=1}^k \int_{-\infty}^{\infty} dy_i P_{a_i}(y_i + 2\pi m_i) P_{\mu(a_i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2}. \quad (9)
\end{aligned}$$

In the same way as 1 instanton sector, using the Taylor series expansion of the Hermite polynomial, we obtain for the above multiple integrals,

$$\begin{aligned}
& \prod_{i=1}^k \int_{-\infty}^{\infty} dy_i P_{a_i}(y_i + 2\pi m_i) P_{\mu(a_i)}(y_i - 2\pi m_i) e^{-\frac{N}{2A} y_i^2} \times (h_{a_1} \dots h_{a_k})^{-1} \\
&= \prod_{l=1}^k \sum_{i_l=0}^{\min(a_l, \mu(a_l))} \frac{\mu(a_l) C_{i_l}}{(a_l - i_l)!} \left(\frac{2\pi m N}{A} \right)^{a_l - i_l} (-2\pi m)^{\mu(a_l) - i_l}. \quad (10)
\end{aligned}$$

The each series in the above equation can be expressed as a contour integral by the following transformation formula.

$$\sum_{i=0}^{\min(a,b)} \frac{b C_i}{(a-i)!} (\alpha)^{a-i} (\beta)^{b-i} = \oint \frac{dt}{2\pi i} e^{\alpha \beta t} \frac{1}{t} \left(\frac{1}{\beta t} \right)^a (\beta(t+1))^b, \quad (11)$$

where the contour of t encircles the origin counterclockwise. Using this formula, we obtain $w(\{m\})$ for k instantons,

$$w(\{m\}) = \frac{Z_G(\frac{N}{2A})}{N(N-1)\cdots(N-k+1)} \sum_{a_1 \neq \dots \neq a_k}^{N-1} \sum_{\mu \in S_k} \text{sgn} \mu \oint \frac{dt_1}{2\pi i} \cdots \oint \frac{dt_k}{2\pi i} \frac{1}{t_1 \cdots t_k} \\ \times e^{-\frac{4\pi^2 N}{A}(m_1^2 t_1 + \dots + m_k^2 t_k)} \left(\frac{m_{\mu(1)}}{m_1} \frac{1+t_{\mu(1)}}{t_1}\right)^{a_1} \cdots \left(\frac{m_{\mu(k)}}{m_k} \frac{1+t_{\mu(k)}}{t_k}\right)^{a_k}. \quad (12)$$

We remark the configurations such as $a_1 = a_2$ do not affect the above equation. Hence we can drive $\sum_{a_1 \neq \dots \neq a_k}^{N-1}$ to independent sum $\sum_{a_1}^{N-1} \cdots \sum_{a_k}^{N-1}$. In that expression there are many pole free terms, then, after eliminating pole free terms, this integrals finally become

$$w(\{m\}) = \frac{Z_G(\frac{N}{2A})}{N(N-1)\cdots(N-k+1)} \oint \frac{dt_1}{2\pi i} \cdots \oint \frac{dt_k}{2\pi i} \\ \times e^{-\frac{4\pi^2 N}{A}(m_1^2 t_1 + \dots + m_k^2 t_k)} \left(1 + \frac{1}{t_1}\right)^N \cdots \left(1 + \frac{1}{t_k}\right)^N \\ \times \sum_{\mu \in S_k} \text{sgn} \mu \frac{1}{\frac{m_{\mu(1)}}{m_1}(1+t_{\mu(1)}) - t_1} \cdots \frac{1}{\frac{m_{\mu(k)}}{m_k}(1+t_{\mu(k)}) - t_k}. \quad (13)$$

The last sum is just determinant of matrix M which has (ij) element

$$M_{ij} \equiv \frac{1}{\frac{m_j}{m_i}(1+t_j) - t_i}. \quad (14)$$

It's diagonal element is 1 and this determinant has all information about the interaction between instantons.

2.3 partition function

We can rewrite the partition function as another contour integral. For this purpose, we must expand $w(\{m\})$ by number of instantons and classify elements of the permutation group S_k into the conjugacy classes. By symmetry argument, the contribution from different elements belonging to same conjugacy class is same. Then we obtain,

$$Z_{QCD} = Z^{\text{weak}} \sum_{n=0}^N \sum_{m_1, \dots, m_n \neq 0} e^{-\frac{2\pi^2 N}{A}(m_1^2 + \dots + m_n^2)} \oint \frac{dt_1}{2\pi i} \cdots \oint \frac{dt_n}{2\pi i} e^{-N(\Phi_{m_1}(t_1) + \dots + \Phi_{m_n}(t_n))} \\ \times \sum_{\text{conj.class}} \text{sgn}[\sigma] T[\sigma] M_{1\sigma_1} M_{2\sigma_2} \cdots M_{n\sigma_n}, \quad (15)$$

where

$$Z^{\text{weak}} \equiv e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} Z_G\left(\frac{N}{2A}\right) = \text{const } e^{N^2(\frac{A}{24} - \frac{1}{2} \log A) - \frac{A}{24}}, \quad (16)$$

$$\Phi_m(t) \equiv \frac{4\pi^2 m^2 t}{A} - \log\left(1 + \frac{1}{t}\right). \quad (17)$$

Z^{weak} is the partiton function in the weak coupling region and it has simple A dependence. We assume σ has cycle structure $[1^{\sigma_1} \cdots n^{\sigma_n}] (\sigma_1 + \cdots + n\sigma_n = n)$ and define $T[\sigma]$ as number of elements in the conjugacy class which σ belongs to,

$$T[\sigma] = \frac{n!}{1^{\sigma_1}\sigma_1! \cdots n^{\sigma_n}\sigma_n!}, \quad (18)$$

$$\text{sgn}\sigma = (-)^{\sigma_2+\sigma_4+\cdots}, \quad (19)$$

$$\sum_{\text{conjclass}} = \sum_{\substack{\sigma_1, \dots, \sigma_n=0 \\ \sigma_1+\cdots+n\sigma_n=n}}. \quad (20)$$

We use (12345)(67) type element as the representative element of the conjugacy class. Because the multiple integrals factorize according to the cycle structure of σ , contribution from the multi-instanton are exponentiated with a constraint $(\sigma_1 + \cdots + n\sigma_n = n)$ which is expressed by another contour integral. We get the following result:

$$Z_{QCD} = Z^{\text{weak}} \oint \frac{dz}{2\pi i} \frac{1}{z^{N+1}} \frac{1}{1-z} \exp\left(\sum_{j=1}^N \frac{(-)^{j-1}}{j} z^j \alpha_j\right), \quad (21)$$

where

$$\begin{aligned} \alpha_j \equiv & \sum_{m_1, \dots, m_j \neq 0} e^{-\frac{2\pi^2 N}{A}(m_1^2 + \cdots + m_j^2)} \oint \frac{dt_1}{2\pi i} \cdots \oint \frac{dt_j}{2\pi i} e^{-N(\Phi_{m_1}(t_1) + \cdots + \Phi_{m_j}(t_j))} \\ & \times M_{12} M_{23} \cdots M_{j1} \end{aligned} \quad (22)$$

is the ‘‘connected’’ amplitude of the j instantons.

Some remark should be noted. In the large N limit, the multiple contour integrals for $w(\{m\})$ or α_j are dominated by the saddle points. Fortunately, it is no need for solving complicated saddle point equations in our case. The saddle point equations are decoupled and the solutions of the equations are copies of the saddle point found by Gross and Matytsin. Then if we neglect the interaction term $M_{12}M_{23} \cdots M_{j1}$, α_j becomes factorized such as $(\alpha_1)^j$ and the behavior change from exponential damping to oscilating at $A = \pi^2 m^2$ observed in [5] still hold for the multi-instanton amplitude. But there is a remarkable feature in the interaction term of the multi-instanton sector.

We point out that in the region of $A > \pi^2$ special charge configurations of the multi-instanton exist. If we consider the following configurations in α_j (even j):

$$m_1 = -m_2 = m_3 = -m_4 = \cdots = m, \quad (23)$$

the large N saddle points for any t_i are same and satisfy the next relation:

$$t_{\pm} = \frac{-1 \pm \sqrt{1 - \frac{A}{\pi^2 m^2}}}{2}, \quad t_+ + t_- + 1 = 0. \quad (24)$$

When $A > m^2\pi^2$, both saddle points must be selected. Moreover when $t_i = t_{\pm}$ and $t_{i+1} = t_{\mp}$, all M_{ii+1} 's have pole and usual $N^{-1/2}$ power law (Gauss integral factor) must be changed. Toy example of such phenomena is the following:

$$\oint \frac{dt}{2\pi i} \exp(-N\phi(t)) \frac{1}{(t-t_c)^n}, \quad (25)$$

where t_c is saddle point of $\phi(t)$. We can approximate this to a principal value integral along the steepest descent line and a contour integral along small semi circle around the saddle point and obtain finite value with order $N^{\frac{n-1}{2}} \exp(-N\phi(t_c))$. We call this type of problem “singular saddle point”. Same power change occurs in the multi-instanton sector. The configuration that all M_{ii+1} have pole is dominant contribution with respect to N in the strong coupling region. We call this phenomena “large N neutrality”. By “neutrality”, we would not intend general neutral configuration which satisfy $\sum_{i=1}^N m_i = 0$, but intend that all instantons have equal absolute charge and number of positive charge instantons and number of negative charge instantons are same. Hence by definition “neutrality” in α_j with odd j is ruled out. Bellow we will see α_j with odd j is in fact negligible in the large N limit.

3 One body and two bodies effective action

Below we will see genereal 2n bodies interactions equally contribute to the phase transition. Even so, it would be worthwhile to analyze familiar two bodies interaction between instantons in detail. We find that the interaction between instantons is sensitive to A and m .

3.1 one body effective action

First we consider the contribution of one instanton with charge m to the partition function. In the large N limit, the saddle point equation $\Phi'_m(t) = 0$ has two solution $t_{\pm} = \frac{-1 \pm \sqrt{1-A/\pi^2 m^2}}{2}$. We see quite different behavior according to $A \gtrless \pi^2 m^2$.

3.1.1 $m^2 > A/\pi^2$

Both saddle points exist on negative real axis and satisfy

$$\Phi_m(t_-) < \Phi_m(t_+). \quad (26)$$

Nevertheless only t_+ saddle point is selected since t_- steepest decent line is not consistent with the original contour. Hence we obtain in the large N limit ,

$$w(m) \simeq (-) \frac{1}{\sqrt{2\pi N |\Phi_m^{(2)}(t_+)|}} e^{-N\Phi_m(t_+)}. \quad (27)$$

If we define $\gamma(x)$ as [5]

$$\gamma(x) \equiv \sqrt{1-x} - \frac{x}{2} \log \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}, \quad (28)$$

we obtain

$$Z_{\text{1inst}} \equiv e^{-\frac{2\pi^2 m^2 N}{A}} w(m) = \frac{(-)^{N-1}}{\sqrt{2\pi N \frac{16\pi^4 m^4}{A^2} \sqrt{1 - \frac{A}{\pi^2 m^2}}}} e^{-\frac{2\pi^2 m^2 N}{A} \gamma(\frac{A}{\pi^2 m^2})}. \quad (29)$$

Since $\gamma(x)$ is positive real for $x < 1$ but pure imaginary for $x > 1$, self energy term survives in this region but disappears at $A = m^2 \pi^2$.

3.1.2 $m^2 < A/\pi^2$

In this case both saddle points are selected because t_{\pm} steepst descent lines are consistent with original contour and satisfy

$$\text{Re}[\Phi_m(t_+)] = \text{Re}[\Phi_m(t_-)]. \quad (30)$$

Therefore, we get

$$Z_{\text{1inst}} = \frac{(-)^N}{\sqrt{2\pi N \frac{16\pi^4 m^4}{A^2} \sqrt{\frac{A}{\pi^2 m^2} - 1}}} (e^{i\frac{3}{4}\pi - \frac{2\pi^2 m^2 N}{A} \gamma(\frac{A}{\pi^2 m^2})} + e^{-i\frac{3}{4}\pi + \frac{2\pi^2 m^2 N}{A} \gamma(\frac{A}{\pi^2 m^2})}). \quad (31)$$

Since $\gamma(x)$ are pure imaginary in this region, the above equation has order $N^{-1/2}$ in the large N limit.

3.2 two bodies effective action

Next we consider the contribution of two instantons with charge m and m' to the partition function in the large N limit.

3.2.1 $m^2 \geq m'^2 > A/\pi^2$

Straight forward application of large N saddle point method gives

$$w(m, m') = \frac{1}{\sqrt{2\pi N |\Phi_m^{(2)}(t_+)|}} \frac{1}{\sqrt{2\pi N |\Phi_{m'}^{(2)}(s_+)|}} e^{-N(\Phi_m(t_+) + \Phi_{m'}(s_+))} \det M(t_+, s_+), \quad (32)$$

where $t_{\pm} = \frac{-1 \pm \sqrt{1 - A/\pi^2 m^2}}{2}$, $s_{\pm} = \frac{-1 \pm \sqrt{1 - A/\pi^2 m'^2}}{2}$ and

$$\det M(t, s) = 1 - \frac{1}{(\frac{m'}{m}(1+s) - t)(\frac{m}{m'}(1+t) - s)}. \quad (33)$$

As a further approximation we assume $m'^2 \gg A/\pi^2$, then we obtain

$$\begin{aligned} Z_{2\text{inst}} &\equiv e^{-\frac{2\pi^2 N}{A}(m^2+m'^2)} w(m, m') \\ &\sim \frac{1}{N} e^{-2N(q^2+q'^2)+N(\log q^2+\log q'^2)+\log(q-q')^2}, \end{aligned} \quad (34)$$

where $q \equiv \sqrt{\pi^2/A} m$ is effective charge and we used the following expansion formula of γ around $x = 0$:

$$\gamma(x) = 1 + \frac{x}{2} \log\left(\frac{x}{4}\right) - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!(n+1)!} \left(\frac{x}{2}\right)^{n+1}. \quad (35)$$

We note that the interaction term is next leading order in $1/N$ and the self energy term strongly suppresses this configuration.

3.2.2 $m^2 \leq m'^2 < A/\pi^2$ and $m \neq -m'$

In the same way we obtain,

$$\begin{aligned} w(m, m') &= \frac{1}{\sqrt{2\pi N|\Phi_m^{(2)}(t_+)|}} \frac{1}{\sqrt{2\pi N|\Phi_{m'}^{(2)}(s_+)|}} e^{i\frac{3}{2}\pi - N(\Phi_m(t_+) + \Phi_{m'}(s_+))} \det M(t_+, s_+) \\ &+ \frac{1}{\sqrt{2\pi N|\Phi_m^{(2)}(t_-)|}} \frac{1}{\sqrt{2\pi N|\Phi_{m'}^{(2)}(s_+)|}} e^{-N(\Phi_m(t_-) + \Phi_{m'}(s_+))} \det M(t_-, s_+) \\ &+ \frac{1}{\sqrt{2\pi N|\Phi_m^{(2)}(t_+)|}} \frac{1}{\sqrt{2\pi N|\Phi_{m'}^{(2)}(s_-)|}} e^{-N(\Phi_m(t_+) + \Phi_{m'}(s_-))} \det M(t_+, s_-) \\ &+ \frac{1}{\sqrt{2\pi N|\Phi_m^{(2)}(t_-)|}} \frac{1}{\sqrt{2\pi N|\Phi_{m'}^{(2)}(s_-)|}} e^{-i\frac{3}{2}\pi - N(\Phi_m(t_-) + \Phi_{m'}(s_-))} \det M(t_-, s_-). \end{aligned} \quad (36)$$

If we further assume $m'^2 \ll A/\pi^2$, then we obtain

$$Z_{2\text{inst}} \sim \frac{1}{N} [\theta(qq') + \theta(-qq') e^{\log(q-q')^2 - \log(q+q')^2}], \quad (37)$$

where we used the following property of γ :

$$\gamma(x) \simeq -\frac{i\pi}{2}x \quad x \gg 1. \quad (38)$$

We note that in this region the self energy term almost disappears and another interaction term $\log(q+q')^2$ is added.

3.2.3 $m^2 < A/\pi^2$ and $m = -m'$

As is already noted, in this case we encounter the singular saddle point.

$$w(m, -m) = \oint \frac{dt}{2\pi i} \oint \frac{ds}{2\pi i} e^{-N(\Phi_m(t) + \Phi_{m'}(s))} \left(1 - \frac{1}{(1+t+s)^2}\right). \quad (39)$$

Since the 1st term of the determinant has no singularity at saddle points, we can neglect this term and obtain

$$w(m, -m) \simeq -N^2 \int_0^\infty dz z e^{-Nz} f(z)^2, \quad (40)$$

where

$$f(z) \equiv \oint \frac{dt}{2\pi i} e^{-(\frac{4\pi^2 m^2}{A} + z)Nt} \left(1 + \frac{1}{t}\right)^N. \quad (41)$$

From above equation $f(z)$ is essentially α_1 with minor changes. If the z is above $4 - 4\pi^2 m^2/A$, $\exp(-Nz/2)f(z)$ has the behavior of exponential damp. Hence we can restrict the z integration range up to $4 - 4\pi^2 m^2/A$. In addition, $\exp(-Nz)f(z)^2$ has both oscillating and constant terms. Oscillating term corresponds to $t = s = t_\pm$ which give order N^{-1} for $\exp(-\frac{4\pi^2 m^2 N}{A})w(m, -m)$, and constant term corresponds to $t = t_\pm, s = t_\mp$ which give order N for $\exp(-\frac{4\pi^2 m^2 N}{A})w(m, -m)$. In the large N limit we obtain

$$Z_{2\text{inst}} \simeq -\frac{2N}{\pi} \left[\frac{\pi}{2} - \arcsin\left(\frac{2\pi^2 m^2}{A} - 1\right) - \frac{2\pi^2 m^2}{A} \sqrt{\frac{A}{\pi^2 m^2} - 1} \right]. \quad (42)$$

4 Numerical calculation

In this section we show the result of numerical calculation of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and free energies for $N=3,4,5$ using the *Mathematica*. We assume $0 < A < 4\pi^2$. From the above argument, in this region we can restrict the instanton charge as $m_i = \pm 1$ (which we call ‘truncated’) in multi-instanton sector.

Fig1,2 show that $N = 10$, $a \equiv A/\pi^2 = 2$ results respectively for $\alpha_1, \alpha_2, \alpha_3$ and α_4 . We observe that compared to the α ’s with $a > 1$, the α ’s with $a < 1$ are negligible. Moreover we see remarkable difference between α_1, α_3 (even j) and α_2, α_4 (odd j) with respect to order and shape in the region of $a > 1$. The shapes of α_2, α_4 are almost same and both have the properties of non-oscillating and linear scaling with respect to N . On the other hand the shapes of α_1, α_3 are different. But both have the properties of oscillating and $N^{-1/2}$ scaling. We already observed that α_1 has $N^{-1/2}$ scaling and α_2 has linear(N) scaling using the large N saddle point method. The numerical calculation indicates the agreement with eq (29),(42). From this observation and the argument of section 2, we conjecture the following scenario of the large N phase transition. That is,

α_j with even j have linear scaling with respect to N in the region of $a > 1$ and inside the exponential in eq (21), we have

$$\sum_{j=1}^N \frac{(-)^{j-1}}{j} z^j \alpha_j \simeq - \sum_{n=1}^{[N/2]} \frac{1}{2n} z^{2n} \alpha_{2n} \simeq \mathcal{O}(N^2). \quad (43)$$

This gives order N^2 result for the free energy in $a > 1$. Thus α_j with even j equally contribute to the large N phase transition.

Fig3 show the truncated free energies

$$F = \frac{1}{N^2} \log \left[\oint \frac{dz}{2\pi i} \frac{1}{z^{N+1}} \frac{1}{1-z} \exp \left(\sum_{j=1}^N \frac{(-)^{j-1}}{j} z^j \alpha_j \right) \right] \quad (44)$$

for $N=3,4$ and 5. The partiton function for $N=5$ turns to have negative value in $a \gtrsim 3.2$. This causes the singular behavior around $a \simeq 3.2$ of the $N = 5$ graph (here we plot real part of the F). We observe that the graphs stand up at $a \simeq 1$ and in the neighborhood of $a = 1$ the graphs seem to converge in the large N limit. But in the region $a \gtrsim 2.5$ discrepancy of the $N=3,4$ and 5 graphs is large. We think the truncation becomes invalid in this region. Hence we may conclude that near $A = \pi^2$ nature of the large N phase transition remain at $N = 3, 4, 5$ level, but we cannot distiguish whether this behavior change is smoose or not.

5 Outlook

We have reformulated the multi-instanton amplitude as the multiple contour integrals and found that this representation make clear the properties and the effect of multi-instanton in two dimensional QCD. In particular “neutral” configurations of the even number instantons are dominated in the large N limit. But we think further investigation is needed about

1. derivation of the order of large N phase transition in terms of our contour integral representation,
2. clearcut order parameter and Ginzburg-Landau effective action,
3. possibility of phase transitions at $A = \pi^2 k^2$ (k :positive integer).

Some explanations are needed.

About point 1, we did not obtain the free energy analytically because we do not know general method for solving the singular saddle point problem. But if such method is found out, we will obtain analytic form of the free energy and hence the order of the phase transition.

The reader may be confused with resspect to point 2. By ‘clearcut’, we intend order parameter which is zero or nonzero according to $A \gtrless \pi^2$. In order to drive argument

of phase transition to Ginzburg -Landau type, we must know such order parameter a priori as variable of effective action. In our case instanton number is naive contender, but it is ruled out. From the expression of instanton amplitudes they are not positive definit and can not have the meaning of probability³ (Boltzman weight). Hence we must consider other order parameter. If such order parameter is found out, the effective action will be the following form near critical point ⁴.

$$\Gamma(\phi) = -(\beta - \beta_c)\phi^2 + g\phi^3 \quad (g > 0) \quad (45)$$

About point 3, we ignore the instanton configuration of $m = \pm 2, \pm 3, \dots$ in section 4 since such configuration give exponential damping factor in the region of $\pi^2 < A < 4\pi^2$. Therefore, same argument leads to that in the region of $k^2\pi^2 < A < (k+1)^2\pi^2$ (k : positive integer) we can ignore the instanton configurations of $m = \pm(k+1), \pm(k+2), \dots$. Hence the multi-instanton configurations which should be included are changed at $A = k^2\pi^2$. Other approaches did not predict phase transitions at $A = k^2\pi^2$, but we think whether the changes lead to some phase transitions or not is still important problem.

We hope this work will shed some lights on the global structure of two dimensional QCD.

Acknowledgements

The auther would like to thank Professor K.Ishikawa for his suggestion and careful reading the manuscript.

³That is why we don't use the word 'instanton condensation'.

⁴In the case of n th order phase transition, $g\phi^3$ term is replaced by $g\phi^{2n/(n-1)}$.

Appendix Multi-instanton effect in Wilson loop

In this appendix we note the multi-instanton effect in Wilson loop. Gross and Matytsin also showed the formula for Wilson loop on a sphere [7]:

$$W_n(A_1, A_2) \equiv \langle \frac{1}{N} \text{tr} U^n \rangle. \quad (46)$$

Here the loop divide the sphere with area A into two disks with area A_1, A_2 and U is gauge field holonomy along the loop .

$$\begin{aligned} & W_n(A_1, A_2) Z_{QCD} \\ = & e^{\frac{A}{24}(N^2-1)} \left(\frac{N}{A}\right)^{N^2} \frac{1}{N} \sum_{k=1}^N \sum_{\{m\}} e^{-\frac{2\pi^2 N}{A} \sum_{i=1}^N m_i^2 - 2\pi i n m_k \frac{A_2}{A} - \frac{n^2 A_1 A_2}{2NA}} \\ \times & \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i \Delta(y_i + 2\pi m_i + \frac{in A_2}{N} \delta_{ik}) \Delta(y_i - 2\pi m_i + \frac{in A_1}{N} \delta_{ik}) e^{-\frac{N}{2A} \sum_{i=1}^N y_i^2}. \end{aligned} \quad (47)$$

From above equation, calculating Wilson loop is almost same as calculating the partition function with additional imaginary charged instanton. Hence in the same way as the partition function, we can rewrite this as multiple contour integrals:

$$\begin{aligned} = & Z^{\text{weak}} \sum_{l=0}^N \frac{1}{l!} \sum_{m_1, \dots, m_l \neq 0} e^{-\frac{2\pi^2 N}{A} (m_1^2 + \dots + m_l^2)} \oint \frac{dt_1}{2\pi i} \dots \oint \frac{dt_l}{2\pi i} \\ \times & e^{-\frac{4\pi^2 N}{A} (m_1^2 t_1 + \dots + m_l^2 t_l)} \left(1 + \frac{1}{t_1}\right)^N \dots \left(1 + \frac{1}{t_l}\right)^N \\ \times & \left[\frac{l}{N} e^{-2\pi i n m_1 \frac{A_2}{A} + 2\pi i n m_1 \frac{A_1 - A_2}{A} t_1 - \frac{n^2}{N} \frac{A_1 A_2}{A} t_1} \det^{(l)} B \right. \\ & \left. + \frac{1}{N} \oint \frac{dt_{l+1}}{2\pi i} e^{-\frac{n^2}{N} \frac{A_1 A_2}{A} t_{l+1}} \left(1 + \frac{1}{t_{l+1}}\right)^N \det^{(l+1)} C \right], \end{aligned} \quad (48)$$

where

$$B_{ij} = \frac{1}{\frac{\beta_j}{\beta_i}(1+t_j) - t_i}, \quad (49)$$

$$C_{ij} = \frac{1}{\frac{\gamma_j}{\gamma_i}(1+t_j) - t_i} \quad (50)$$

and

$$\beta_i = -2\pi m_i + \frac{in}{N} A_1 \delta_{i1}, \quad (51)$$

$$\gamma_i = -2\pi m_i \quad (\text{for } i = 1, \dots, n), \quad (52)$$

$$\gamma_{n+1} = \frac{in}{N} A_1. \quad (53)$$

Using this formula we can calculate multi- instanton effect of the Wilson loop in the large N limit.

For example in 0 instanton sector (weak coupling phase), we obtain,

$$W_n(A_1, A_2)Z_{QCD}|_{0\text{inst}} = Z^{\text{weak}} \frac{1}{N} \oint \frac{dt}{2\pi i} e^{-\frac{n^2 A_1 A_2}{NA} t} (1 + \frac{1}{t})^N. \quad (54)$$

By rescaling $t \rightarrow Nt$, we obtain the known result:

$$W_n(A_1, A_2) \simeq \frac{1}{n} \sqrt{\frac{A}{A_1 A_2}} J_1(2n \sqrt{\frac{A_1 A_2}{A}}). \quad (55)$$

References

- [1] D.Gross , Nucl.Phys.**B400**(1993) 161;
D.Gross and W.Taylor , Nucl.Phys.**B400** (1993) 181;
D.Gross and W.Taylor , Nucl.Phys.**B403** (1993) 395.
- [2] M.Douglas and V.Kazakov , Phys.Lett.**B319** (1993) 219.
- [3] E.Witten, Commun. Math. Phys. **141** (1991) 153 ; J. Geom. Phys. **9** (1992) 303.
- [4] A.Minahan and Polychronakos , Nucl.Phys.**B422** (1994) 172.
- [5] D.Gross and A.Matytsin , Nucl.Phys.**B429** (1994) 50.
- [6] M.Caselle, A.D’Adda, L.Magnea and S.Panzeri , in: Proc. 1993 Trieste Workshop on High Energy Physics and Cosmology.
- [7] D.Gross and A.Matytsin , hep-th/9410054.
- [8] W.Taylor , hep-th/9404175.
- [9] A.Migdal , Sov.Phys.JETP **42** (1975) 413;
B.Rusakov , Mod.Phys.Lett. **A5** (1990) 693.

Figure captions

Fig 1: The truncated α 's for $N = 10$ and various $a \equiv A/\pi^2$ are shown.

Fig 2: The truncated α 's for $a = 2$ and various N are shown.

Fig 3: The truncated free energies for $N = 3, 4, 5$ and various a are shown.

Fig3: The truncated Free Energies for N=3,4,5

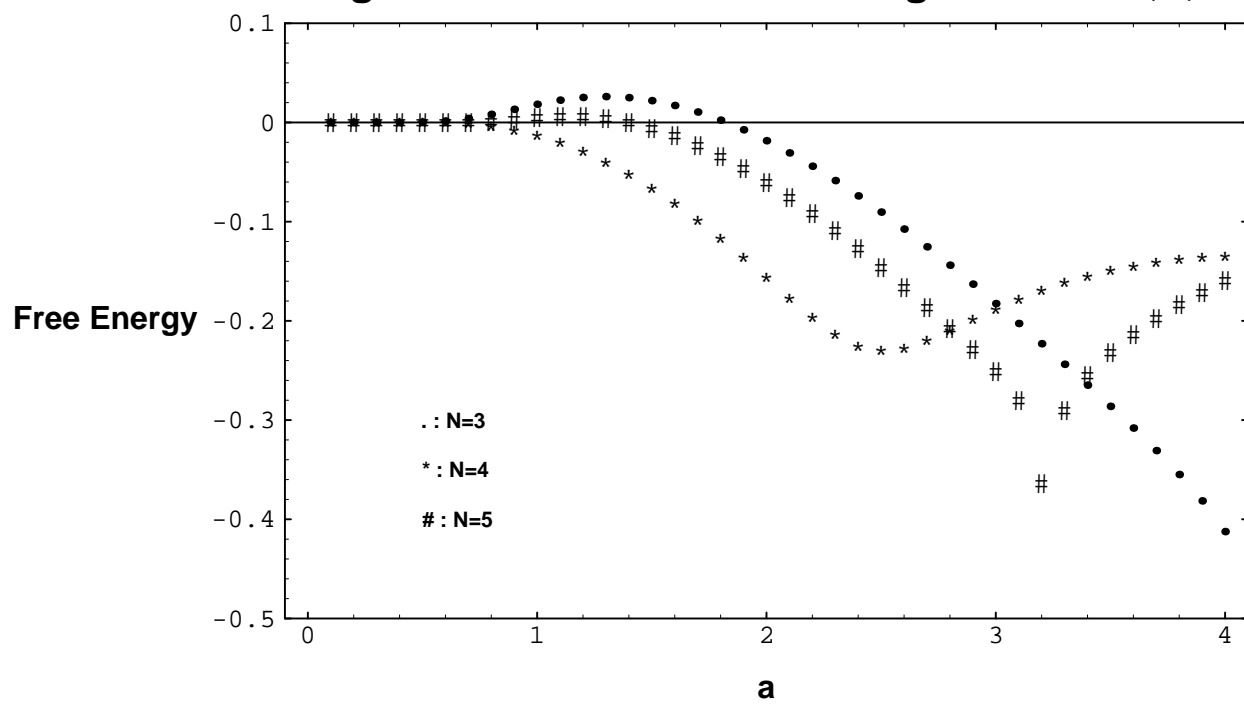


Fig1: The truncated ALPHA's for N=10

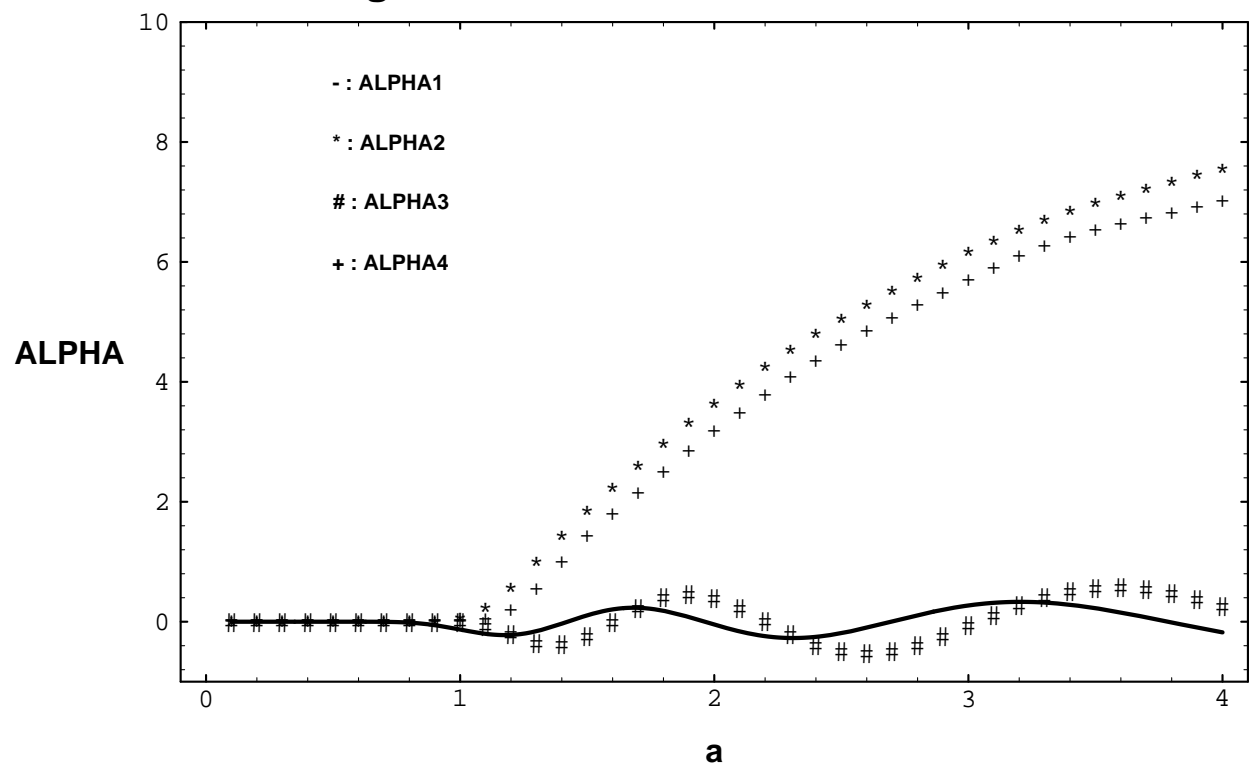


Fig2:The truncated ALPHA's for a=2

